Algebraic Combinatorics

A Pieri Rule for Skew Shapes

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Self Reflection

When I was first offered the opportunity to participate in this independent study in algebraic combinatorics, I knew it would be tough, but with the right amount of time devoted, I would be able to push through. However, after reading my first page in Richard Stanely's "Enumerative Combinatorics", I realized I undermined the difficulty of this course. After 2 hours of re-reading the same page over and over, sentence by sentence, I had no idea what was going on. I knew I had to resort to something else: Google. So the next day, I spent several hours on Google, watching videos and reading different articles. However, after several hours, I faced the same dillemna: I didn't understand anything the book was trying to explain. I tried the same process day after day, eventually leading up to our first meeting with Professor Anna Pun. When she asked us how much we were able to understand this week, I looked around the room looking for a savior that would have the same answer as me: nothing! Luckly, I wasn't the first person to share my thoughts. After some thoughts from others, we came to to realization that none of us was able to make significant progress in the book.

Luckily, Professor Pun was there to help us overcome these challenges. Through her guidance and feedback weekly, I was able to develop different strategies to learn math on my own. First and foremost, she taught us examples! Examples, examples, and more examples. One thing she emphasized was the more abstract something gets, the more examples I should create. A key realization I came to was that everything was initially extremely difficult because I was looking at lambdas, mus, alphas, etc. However, once I made those into integers that completely changed my understanding. As a result, in following weeks, I dedicated significant time to creating my own example for every equation or theorem the textbook gave me. A key concept to creating examples I discovered is to start small. Professor Pun taught us to start as small as possible and keep increasing until you understand the pattern. At the end of the day that is what it come down to. Now I am comfortable using a variety of resources, including textbooks, online lectures, and academic papers, because I understood how to learn from them.

Lastly, I would say engaging in weekly discussions with my peers gave me support and encouragement to keep going. Every one of us expressing our challenges and solutions to overcome obstacles was extremely beneficial. Along with that, just knowing that fellow students are going through the same journey as me, reminded me that if they could do it, I could do it too.

In conclusion, embarking on this independent study in algebraic combinatorics has been a life-changing journey filled with struggles, but at the same time lots of learning. While the initial hurdles were very tough, they served as stepping stones for personal and intellectual growth. Moving forward, I am now comfortable and excited to continue studying math independently without having a Professor just hand me the answers.

Introduction to Schur Polynomials

Schur polynomials are fundamental objects in algebraic combinatorics. In general, Schur polynomials are symmetric functions that provide a basis in the space of symmetric functions and are represented as S_n. A symmetric function is a function that remains unchanged under all permutations of their variables. In other words, if you swap the order of variables in a symmetric function, the function does not change.

Symmetric Function

For instance, consider the following example:

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2$$

where x_1, x_2, x_3 are variables

This function is symmetric because swapping any two x variables does not change the function. For example, if we swap x_1 and x_3, we get:

$$f(x_3, x_2, x_1) = x_3^2 + x_2^2 + x_1^2$$

which is the same as the original function, just in different order.

Schur functions are symmetric functions that are defined using two different approaches: algebraically and combinatorically. For the purposes of this paper, we will only focus on the combinatorical definition of schur functions. Lets start by first defining a Young Tableaux.

Young Tableaux

A Young tableau is a combinatorial object used to represent various symmetric functions along with other algebraic structures. It is represented by a tableau, which you can think of as a table with cells arranged in rows and columns, typically filled with positive integers, but you could fill them with anything.

A Standard Young tableau is a special Young tableau where the entries of positive integers in each row and each column are strictly increasing from left to right and from top to bottom. Strictly increasing just means increasing without repeating. For example, the following is a Standard Young tableau of shape $(3, 2, 1)$:

1	2	3
4	5	
6		

Notice (3, 2, 1) is a partition, often denoted as λ, of 6 and correlates to the number of cells in each row of the Tableau.

Semistandard Young Tableaux

When it comes to Schur functions, we consider semi-standard Young Tableaux, abbreviated as SSYT. A semi-standard Young tableau is a Young tableau where the entries in each column are strictly increasing from top to bottom, and the rows are weakly increasing. Weakly increasing just means increasing with repeating values allowed. Here's an example of a semistandard Young tableau with $\lambda = (3, 2, 1)$:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$$

Notice in this tableau, the first row is weakly increasing because 1 appears twice, but columns don't have repeated entries.

Defining a Schur Function using a SSYT

Choose λ and a bound n on the size of the entries in each semi-standard tableau T. Let $x^T = \prod_{i=1}^{n} x_i^j$, where j is the number of i's in T. Then the Schur polynomial is

$$s_\lambda(x_1, \ldots, x_n) := \sum_{\text{semistandard } T} x^T.$$

That just means, given a partition λ and the number of variables n, the Schur function for that partition can be found by adding all the possible semi-standard tableau with n being the largest integer used.

Example Let $\lambda = (3, 2)$. Then the list of possible semistandard tableaux of shape λ where $n = 3$ are:

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 3 & 3 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 & 3 \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}$$

$$x_1^3x_2^2 + x_1^3x_2x_3 + x_1^3x_3^2 + x_1^2x_2^3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2$$

$$\begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 2 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline 3 & 3 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 3 & 3 & \\ \cline{1-2}\end{array} \quad \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 & 3 & \\ \cline{1-2}\end{array}$$

$x_1^2x_2^2x_3 \quad + x_1^2x_2x_3^2 \quad + x_1^2x_3^3 \quad + x_1x_2^3x_3 \quad + x_1x_2^2x_3^2 \quad + x_1x_2^2x_3^2$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 3 & 3 & \\ \cline{1-2}\end{array}$$

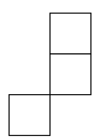
$+ x_1x_2x_3^3$

Notice how we multiplied all variables (cells) in a tableau, and therefore got the degree of each term to be the number of cells – 5. This directly comes from $\lambda = (3, 2)$, which is a partition of 5. When we add the tableau together, we get our final result:

$$s_{32}(x_1, x_2, x_3) = x_1^3x_2^2 + x_1^3x_2x_3 + x_1^3x_3^2 + x_1^2x_2^3 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^2x_2^2x_3 + x_1^2x_2x_3^2 + x_1^2x_3^3 + x_1x_2^3x_3 + x_1x_2^2x_3^2 + x_1x_2^2x_3^2 + x_1x_2x_3^3$$

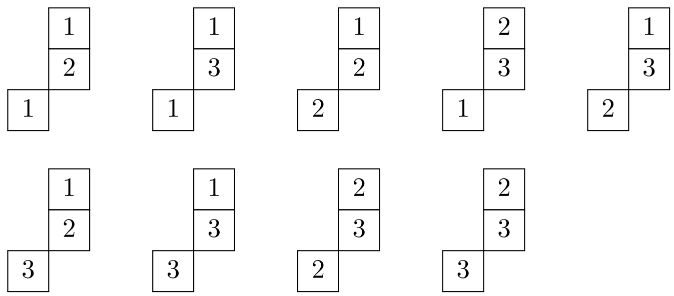
Skew Schur Function using a SSYT

Now since we have a better understanding of Schur functions, let's understand skewed shapes. A skewed shape is denoted as λ/μ where $\mu_i \leq \lambda_i$ for all i. That just means the shape of μ is contained within λ. Something to note is that partition λ is secretly just the skew shape $\lambda/\emptyset$. For instance, the shape (2, 2, 1)/(1, 1) has λ = 2, 2, 1 and μ = 1, 1. The Young Tableau for this skewed partition is:



Notice how this is tableau of $\lambda = (2, 2, 1)$ with the first two cells in the first row and the first cell in the second row missing. That is due to $\mu = (1, 1)$

Replacing λ by λ/μ in the definition of the Schur function I gave earlier gives us the definition of the skew Schur function $s_{\lambda/\mu}$. Now the sum is going to be over all SSYTs of shape λ/μ. For example, the skewed shape $\lambda/\mu = 221/11$ gives us the possible young tableaux:



Now we can find the Schur Polynomial by adding all these tableaux:

$s_{221/11} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + 3x_1x_2x_3 + x_2^2x^3 + x_1x_3^2 + x_2x_3^2$

Pieri Rule

The Pieri Rule gives us an easy way to compute the product of two schur functions $s_\lambda s_n$.

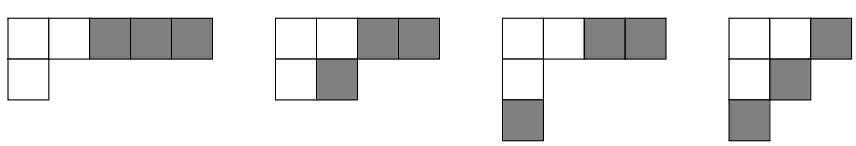
Theorem: For any partition λ and positive integer n:

$$s_\lambda s_n = s_\lambda h_n = \sum_{\lambda^+/\lambda} s_{\lambda^+},$$

That is just the sum of all SSYTs where λ^+/λ is a horizontal strip of size n. A horizontal strip is wherever there are not two cells in the same column. A horizontal strip of size n, just means we need to add n boxes to the SSYT such that no two added boxes are in the same column. For example, lets take

$$s_{21}s_3 = s_{21}h_3 = \sum_{\lambda^+/\lambda} s_{\lambda^+},$$

We need to find all tableaux in which we could add a horizontal strip of length 3 to the shape (2, 1):



s_{51} $\quad + s_{42}$ $\quad + s_{411}$ $\quad + s_{322}$

Hence, we get:
$s_{21}s_3 = s_{21}h_3 = s_{51} + s_{42} + s_{411} + s_{322}$

Pieri Rule on Skew Shapes

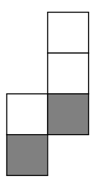
Now we come to our main result. In order to multiply any skew shape λ/μ Schur function and positive integer n, we have

$$s_{\lambda/\mu}s_n = s_{\lambda/\mu}h_n = \sum_{k=0}^{n}(-1)^k \sum_{\substack{k=0 \\ \lambda^+/\lambda \text{ horizontal} \\ \mu/\mu^- \text{ vertical}}}^{n} s_{\lambda^+/\mu^-}$$

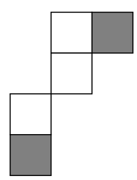
The second sum is over all partitions λ^+ and μ^- such that λ^+/λ is a horizontal strip of size $n-k$ and μ/μ^- is a vertical strip of size k. What that basically means is for every value of k until n, we will counting all the possible SSYT while removing a vertical strip of size k and a horizontal strip of size n-k. For example lets take:

$$s_{221/11}s_2 = s_{221/11}h_2 = \sum_{\lambda^+/\lambda} s_{\lambda^+},$$

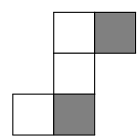
First we need to find all tableaux in $k = 0$ meaning with a horizontal strip of length 2 to the shape (2, 2, 1):



$s_{2211/11}$

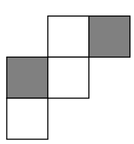


$+\ s_{3211/11}$

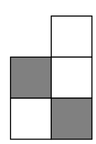


$+\ s_{322/11}$

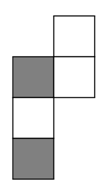
Next we need to find all tableaux in $k = 1$ meaning with a horizontal strip of length 1, and a vertical strip of size 1 to the shape (2, 2, 1):



$s_{321/1}$

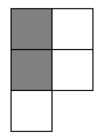


- $s_{222/1}$



- $s_{2211/11}$

Lastly, we need to find all tableaux in $k = 2$ meaning with a vertical strip of length 2 to the shape (2, 2, 1):



$$+s_{222}$$

If we add all our tableaux together we get:
$s_{221/11}s_2 = s_{2221/11} + s_{3211/11} + s_{322/11} - s_{222/1} - s_{2211/1} - s_{321/1} + s_{221}$

We now understand how to multiply a skew-shaped Schur function $s_{\lambda/\mu}$ with s_n. Further research could be dedicated to understanding how to multiply a skew-shaped Schur function $s_{\lambda/\mu}$ with another skew-shaped Schur function $s_{\lambda/\nu}$. I am extremely grateful to Professor Ying Anna Pun for helping me understand this topic and for her willingness to always assist me when I got lost.

Reference: Sami H. Assaf, Peter R. W. Mcnamara. A Pieri rule for skew shapes. 22nd International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2010), 2010, San Francisco, United States. pp.133-144